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The Application of Transportation Algorithm with Volume Discount on Distribution Cost

(A case study of Port Harcourt flour mills Company Ltd.)

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Abstract
Until recently, heavy trucks could load up to any capacity without limit. These trucks normally exceed the average loading capacity of the truck. This was partially due to high transportation cost. Drivers and transport owners together with transport users had to find a way of compensating for the high cost of transport by increasing the truck load so as to maximize profit. This had ripple effect on the state as a whole: increased road accidents, destruction of roads and longer time being spent on the road before getting to destination. There is also the effect of increased cost of goods thereby increasing inflation. This drives the attention of the government to find a lasting solution to the problems. The government therefore went into agreement with transport owners to determine maximum loading capacity of trucks. The purpose of this paper is to find out whether giving discounts on transportation charges could minimize total transportation cost thereby increasing total revenue of both producers and retailers. This study is focused on the Application of Transportation Algorithm with volume Discount on distribution cost using Portharcourt flour mills company plc. This paper is intended to determine the quantity of Golden Penny Flour (in 50kg bags), Golden Penny Semovita (in 10kg bags) and Wheat Offals (also in 50kg bags) that porthacourt flour mills company should distribute in a month in order to minimize transportation cost and maximize profit. A problem of this nature was identified as a Nonlinear Transportation Problem (NTP), formulated in mathematical terms and solved by the KKT optimality condition for the NTP. Thus, analysis revealed the following allocations: 10000 bags of Golden Penny Flour should be supplied to Aba, 7000 of the same product should be supplied to Bayelsa. Allocate 2000 bags of Wheat Offals to Aba, and 9000 bags of the same product to Enugu. Finally, allocate 4000 bags of Golden Penny Semovita to Enugu and 11000 bags of the same product to Onitsha. Hence, the total minimum transportation cost for the above distribution is ₦394,000.

Keywords: transportation algorithm, nonlinear transportation problem (NTP), KKT optimality condition, total minimum transportation cost

INTRODUCTION

When considering transportation, various considerations are apparent. This consideration includes port selection, inland movement, and port to port carrier selection and delivery movement. In addition to these transportation concerns, distribution-related considerations must also be given attention to such as packing/packaging, transit insurance, terms of sale, import duties, handling/loading and method of financing. Nevertheless, even freight companies projecting large volume movements can encounter serious transportation problem in organizing for distribution.

Understanding these transportation problems especially that affects shipping costs is critical. Volume discount, more specifically, targets shipping cost and in minimizing the latter, volume discount must be acquired. Network models and integer programs are well known variety of decision problems. A very useful and widespread area of application is the management and efficient use of scarce resources to increase productivity. These applications include operational problems such as the distributions of goods, production scheduling production and machine sequencing and planning problems such as capital budgeting facility allocation, portfolio selection, and design problems such as telecommunication and transportation network design.

The transportation problem which, is one of network programming problems is a problem that deals with distributing any commodity from any group of 'sources' to any group of destinations or 'sinks' in the most cost effective way with a given 'supply' and 'demand' constraints . Depending on the nature of the cost function, the transportation problem can be categorized into linear and nonlinear transportation problem.

In the linear transportation problem (ordinary transportation problem) the cost per unit commodity shipped from a given source to a given destination is constant, regardless of the amount shipped. It is always supposed that the mileage (distance) from every source to every destination is fixed. To solve such transportation problem we have the streamlined simplex algorithm which is very efficient. However, in reality, we can see at least two cases that the transportation problem fails to be linear.

First, the cost per unit commodity transported may not be fixed for volume discounts sometimes are available for large shipments. This would make the cost function either piecewise linear or just separable concave function. In this case the problem may be formulated as piecewise linear or concave programming problem with linear constraints.

Second, in special conditions such as transporting emergency materials when natural calamity occurs or transporting military during war time, where carrying network may be destroyed, mileage from some sources to some destination are no longer definite. So the choice of different measures of distance leads to nonlinear (quadratic, convex …) objective function.

In both the above cases solving the transportation problem is not as simple as that of the linear one. In this work, solution procedures to the generalized transportation problem taking nonlinear cost function are investigated. In particular, the nonlinear transportation problem considered in this thesis is stated as follows;

- We are given a set of n sources of commodity with known supply capacity and a set of m destinations with known demands.
- The function of transportation cost, nonlinear, and differentiable for a unit of product from each source to each destination.
- We are required to find the amount of product to be supplied from each source (may be market) to meet the demand of each destination in such a way as to minimize the total transportation cost.

Our approach to solve this problem is applying the existing general nonlinear programming algorithms to it making a suitable modification in order to use the special structure of the problem.

STATEMENT OF THE PROBLEM

The prices of commodities are determined by a number of factors; the prices of raw materials, labour, and transport. When price of raw materials increase, so does the price of the commodity. Transportation cost also affects the pricing system. It is assumed that the cost of goods per unit shipped from a given source to a given destination is fixed regardless of the amount shipped. But in reality, the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal cost of shipping one unit might follow a particular pattern.

This paper therefore seeks to develop a mathematical model using optimization techniques to bridge the gap between demand and supply by discounting so as to minimize total transportation cost. The problem that will be addressed in this study centers on the transportation problems experienced by freight companies. Volumes of goods to be shipped incur costs hence acquiring volume discounts could effectively lead to reduced shipping costs.

The key question to be answered is: How freight companies could improve their total output through effectively reducing shipping costs through volume discounts.

LIMITATIONS OF THE STUDY

This study is limited to the Port Harcourt Flour Mills Company ltd, Rivers State, Nigeria. The result of the study could be replicated in other company.

OBJECTIVES OF THE STUDY

The main aim of this study is to design mathematical programme that would improve the total output of freight companies especially since they deal with shipping of goods by volume. Whether maximum profit will be realized with discounts on large volumes means to determine the best transportation route that would lead to low transportation cost and the effective transportation of these goods. We will also provide algorithms and different solution procedures to the different cases that might arise.

TRANSPORT COSTS

Transport Costs and Rates

Transport systems face requirements to increase their capacity and to reduce the costs of movements. All users (e.g. individuals, enterprises, institutions, governments.) have to negotiate or bid for the transfer of goods, people, information and capital because supplies, distribution systems, tariffs, salaries, locations, marketing techniques as well as fuel costs are changing constantly. There are also costs involved in gathering information, negotiating, and enforcing contracts and transactions, which are often referred as the cost of doing business. Trade involves transactions costs that all agents attempt to reduce since transaction costs account for a growing share of the resources consumed by the economy (Shetty; 1959).

Frequently, enterprises and individuals must take decisions about how to route passengers or freight through the transport system. This choice has been considerably expanded in the context of the production of lighter and high value consuming

goods, such as electronics, and less bulky production techniques. It is not uncommon for transport costs to account for 10% of the total cost of a product. Thus, the choice of a transportation mode to route people and freight within origins and destinations becomes important and depends on a number of factors such as the nature of the goods, the available infrastructures, origins and destinations, technology, and particularly their respective distances. Jointly, they define transportation costs.

Transport costs are a monetary measure of what the transport provider must pay to produce transportation services. They come as fixed (infrastructure) and variable (operating) costs, depending on a variety of conditions related to geography, infrastructure, administrative barriers, energy, and on how passengers and freight are carried.

TRANSPORTATION COST ANALYSIS

A typical application of the transportation problem is to determine an optimal plan for shipping goods from various sources to various destinations given supply and demand constraints in order to minimize total shipping cost. It is assumed that the cost of goods per unit shipped from a given source to a given destination is fixed regardless of the amount shipped. However, in actuality the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal cost of shipping one unit might follow a particular pattern.

A transportation service incurs a number of costs: labor, fuel, maintenance etc. this cost can be divided into two: those cost that vary with services or volumes called variable cost and those that do not vary with services called fixed cost.

LINEAR AND NONLINEAR

Transportation Review

The transportation problem was formalized by the French mathematician Gaspard Monge in (1781). Major advances were made in the field during World War II by the Soviet/Russian mathematician and economist Leonid Kantorovich. Consequently, the problem as it is now stated is sometimes known as the Monge-Kantorovich transportation problem as reported by www.historyofmathematics.com.

The problem with the production capacity of each source fixed with constant unit transportation cost was originally formulated by Hitchcock (1941) and was subsequently dealt with independently by Koopmans during Second World War. Analytical solution to this problem has been given by several authors. Stringer and Haley have developed a method of solution using mathematical analogue.

George Dantzig (1951) adapted the simplex method to solve the transportation problem formulated earlier by Hitchcock and Koopmans. Abraham Charnes and William Cooper (1954) derived an intuitive presentation of Dantzig's procedure called the stepping-stone method which follows the basic logic of the simplex method but avoids the use of the tableau and the pivot operations required to get the inverse of the basis.

Martins, Lucia, Craveirinha (2005) presented a study of a bi-dimensional dynamic routing model for telecommunications network. The model uses heuristic methods to solve instability problems. The routing methods through heuristics are compared with the discrete-event simulation in the dynamic routing system. The branch and bound algorithm approach is based on using a convex approximation to the concave cost functions. It is equivalent to the solution of a finite sequence of transportation problems. The algorithm was developed as a particular case of the simplified algorithm for minimizing separable concave functions over linear polyhedral as Falk and Soland. Piece-wise linear over approximation is also the other approach to solve the nonlinear concave transportation problem.

Caputo et al (1991) presented a methodology for optimally planning long-haul road transport activities through proper aggregation of customer orders in separate full-truckload or less-than-truckload shipment in order to minimize total transportation cost. They have demonstrated that evolutionary computation technique may be effective in tactical planning of transportation activities. The model shows that substantial savings on overall transportation cost may be achieved adopting the methodology in a real life scenario.

Zangiabadi and Maleki (2007) presented a fuzzy goal programming approach to determine an optimal compromise solution for the multi-objective transportation problem by assuming that each objective function has a fuzzy goal. A special type of non-linear (hyperbolic) membership function is assigned to each objective function to describe each fuzzy goal. The approach focuses on minimizing the negative deviation variables form 1 to obtain a compromise solution of the multi-objective transportation problem.

Surapati and Roy (2008) presented a priority based fuzzy goal programming approach for solving a multi-objective transportation problem with fuzzy coefficients. Firstly, they defined the membership functions for the fuzzy goals. Subsequently, they transformed the membership functions into membership goals, by assigning the highest degree (unity) of a membership function as the aspiration level and introducing deviational variables to each of them. In the solution process, negative deviational variables are minimized to obtain the most satisfying solution.

Lau et al. (2009) presented an algorithm called the fuzzy logic guided non-dominated sorting genetic algorithm to solve the multi-objective transportation problem that deals with the optimization of the vehicle routing in which multiple depots, multiple customers, and multiple products are considered. Since the total traveling time is not always restrictive as a time constraint, the objective considered comprises not only the total traveling distance, but also the total traveling time.

Lohgaonkar and Bajaj (2010) used fuzzy programming technique with linear and non-linear membership function (hyperbolic, exponential) to find the optimal compromise solution of a multi-objective capacitated transportation problem.

Having reviewed some past researcher's work, we shall now study the application of transportation algorithm with volume discount on distribution cost, using Portharcourt Flour Mills Company LTD as a case study.

RESEARCH METHODOLOGY

A typical application of the transportation problem is to determine an optimal plan for shipping goods from various sources to various destinations given supply and demand constraints in order to minimize total distributing cost. It is assumed that the cost of goods per unit shipped from a given source to a given destination is fixed regardless of the amount shipped. But in actuality the cost may not be fixed. Volume discounts are sometimes available for large shipments so that the marginal cost of distributing one unit might follow a particular pattern. When volume discounts are offered, the objective function or the constraint functions assume a nonlinear form. We therefore use the nonlinear method of solution to solve such a problem using Port Harcourt Flour Mills Company ltd, in Rivers State of Nigeria.

THE KARUSH-KUHN-TUCKER (KKT) OPTIMALITY CONDITION FOR NONLINEAR PROGRAMMING PROBLEM

Given the non linear programming problem:

(NNP) min f(x)

s.t. $g_i(x) \le 0$ i =1, …, k … (1)

$h_j(x) = 0$ j = 1, …, *l*

KARUSH-KUHN-TUCKER NECESSARY OPTIMALITY CONDITIONS

Theorem 3.1.1: Given the objective function $f: \mathbb{R}^n \to \mathbb{R}$ and the constraint function are $g_i: \mathbb{R}^n \to \mathbb{R}$ and $h_j: \mathbb{R}^n \to \mathbb{R}$ and $I = \{i: g_i(x^*) = 0\}$. In addition, suppose they are continuously differentiable at a feasible point x^* and $\nabla g_i(x^*)$ for $i \in I$ and $\nabla h_j(x^*)$ for j = 1, …, *l* be linearly independent. If x^* is minimizer of the problem (NPP), then there exist scalars $\bar{\lambda}_i$; i = 1, …, k and $\bar{\mu}_j$; j =1, …, *l*, called Lagrange multipliers, such that

$$\nabla f(x^*) + \sum_{i=1}^{k} \bar{\lambda}_i \nabla g_i(x^*) + \sum_{j=1}^{l} \bar{\mu}_j \nabla h_j x^* = 0$$

$$\bar{\lambda}_i g_i(x^*) = 0 \quad \ldots \qquad (2)$$

$$\bar{\lambda}_i \ge 0; \quad \bar{\mu}_j \in \mathbb{R}$$

KARUSH-KUHN-TUCKER SUFFICIENT OPTIMALITY CONDITIONS FOR CONVEX NPP

Further, if f and each g_i are convex, each h_j is affine, then the above necessary optimality conditions will be also sufficient.

Justification

Let x be any feasible point different form x*. From the first KKT conditions we obtain

$$\nabla f(x^*)(x - x^*) = -$$

$$\left(\sum_{i=1}^{k} \bar{\lambda}_i \nabla g_i^t(x^*)(x - x^*) + \sum_{j=1}^{l} \bar{\mu}_j \nabla h_j(x^*)(x - x^*) \right)$$

Since each $g_i(x)$ is convex, $\bar{\lambda}_i \ge 0$ and $\nabla h_j(x^*)$ $(x - x^*) = 0$ $\forall j$, we also have

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla g_i^t(x^*)(x - x^*) \le \sum_{i=1}^{k} \bar{\lambda}_i \, [g_i(x) - g_i(x^*)]$$

$$\Rightarrow \qquad \nabla f(x^*)(x - x^*) \ge -\sum \bar{\lambda}_i g_i(x) \ge 0$$

From convexity of f(x), therefore, we get

$f(x) - f(x^*) \ge 0$

$\Leftrightarrow$ $f(x^*) \le f(x)$ for any feasible x.

LINEAR TRANSPORTATION PROBLEM

Transportation Model Problem

Transportation is an example of network optimization problem. It deals with the efficient distribution (transportation) of product (goods) and services from several supply locations (sources) with limited supply, to several demand locations (destinations) with a specified demand with the objective of minimizing total distribution cost; a typical example of which this thesis represents (in analogy).

This objective is achieved under the following constraints;

1. Each demand point receives its requirement.
2. Distributions from supply points do not exceed its available capacity.

This goal is achieved contingent on availability and requirements constraints. Transportation problem therefore assumes that the transportation cost on a given route is directly proportional to the number of units of the commodity transported (Inyama; 2007).

MODEL FORMULATION

The formulation of the transportation model employs double – subscripted variables of the form x_{ij}. Thus, the general formulation of the transportation problem with n sources and m destinations, is given by

$$\text{Minimize } \sum_{i=1}^{n}\sum_{j=1}^{m} C_{ij}x_{ij}$$

$$\text{Subject to } \sum_{i=1}^{n} s_i = s \quad i = 1,2,3,\ldots,n \qquad \ldots (3)$$

$$\sum_{j=1}^{m} d_j = d \quad j = 1,2,3,\ldots,m$$

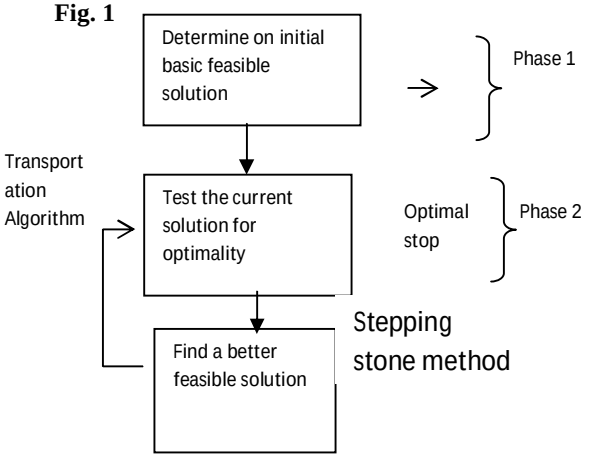
FINDING INITIAL FEASIBLE SOLUTION TO TRASPORTATION PROBLEM

The general formulation of the transportation problem reveals that m supply constraints and n demand constraints translate into m + n total constraints. In the transportation problem however, one of the constraints is redundant resulting in the fact that if, in a balance condition,

$$\sum_{i=1}^{n} s_i \geq \sum_{j=1}^{m} d_j$$

m + n constraints are met then m + n equations will also be met. Only m + n – 1 independent equation, thus, exist and so the initial solution will have only m+ n -1 basic variables. The flow chart below illustrates the various phases leading to the optional solution of a transportation problem

Fig. 1



TRANSPORTATION TABLEAU

The transportation tableau is a unique tabular representation of the transportation problem. The x_{ij} variable gives the number of units transported from source i to destination j (which is to be solved for) while the unit cost for the transportation from i to j, denoted by C_{ij}, is recorded in a small box in the upper – right – hand corner of each cell. Below is the form of the general transportation tableau.

Table 1: Transportation Tableau

To (i) From (j)		DESTINATIONS						
		1	2	…	j	…	m	Supply
SOURCES	1	C_{11} x_{11}	C_{12} x_{12}	…	C_{1j} x_{1j}	…	C_{1m} x_{1m}	S_1
	2	C_{21} x_{21}	C_{22} x_{22}	…	C_{2j} x_{2j}	…	C_{2m} x_{2m}	S_2
	…	…	…	…	…	…	…	…
	i	C_{i1} x_{i1}	C_{i2} x_{i2}	…	C_{ij} x_{ij}	…	C_{im} x_{im}	S_i
	…	…	…	…	…	…	…	…
	n	C_{n1} X_{n1}	C_{n2} X_{n2}	…	C_{nj} X_{nj}	…	C_{nm} x_{mn}	S_n
Demand		d_1	d_2	…	d_j	…	d_m	$\sum S_i = \sum d_j$

METHODS FOR FINDING INITIAL BASIC FEASIBLE SOLUTIONS

The first phase of the solving a transportation problem for optimal solution involves finding the initial basic feasible solution. An initial feasible solution is a set of arc flows that satisfies each demand requirement without supplying more from any origin node than the supply available. Heuristic, a common – sense procedure for quickly finding a solution to a problem is a producer most employed to find an initial feasible solution to a transportation problem. This paper examines three of the more popular heuristics for developing an initial solution to transportation problem.

i. The Northwest corner method
ii. The Least Cost Method
iii. The Vogel's Approximation Method

SOLUTION PROCEDURES TO THE NONLINEAR TRANSPORTATION PROBLEM (NTP)

In this section, we consider a transportation problem with nonlinear cost function. We try to find different solution procedures depending on the nature of the objective function.

Before going to the different special cases, let's formulate the KKT condition and general algorithm for the problem.

Given a differentiable function $C : \mathbb{R}^{nm} \to \mathbb{R}$.
We consider a nonlinear transportation problem (NTP)

$$\min C(x)$$
$$\text{s.t.} \quad Ax = b \quad \ldots \quad (4)$$
$$x \geq 0$$

where

$$x = \begin{pmatrix} x_{11} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{nm} \end{pmatrix}; \quad b = \begin{pmatrix} s_1 \\ \vdots \\ s_n \\ d_1 \\ \vdots \\ d_m \end{pmatrix};$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & & & & & & & & & \\ & & & & 1 & 1 & 1 & 1 & & & & & \\ & & & & & & & & 1 & 1 & 1 & 1 & \dots \\ \vdots & & & \vdots & & & & \vdots & & & & & \\ 1 & & & & 1 & & & & 1 & & & & \\ & 1 & & & & 1 & & & & 1 & & & \\ & & 1 & & & & 1 & & & & 1 & & \\ & & & 1 & & & & 1 & & & & 1 & \dots \\ & & & \vdots & & & & \vdots & & & & \vdots & \ddots \end{pmatrix}$$

The KKT Optimality Condition for the NTP
The transportation table is given as:

$\frac{\partial C(\bar{x})}{\partial x_{11}}$	…	…	…	$\frac{\partial C(\bar{x})}{\partial x_{1m}}$	s_1	u_1
…	…	…	…	…	…	…
…	…	$\frac{\partial C(\bar{x})}{\partial x_{ij}}$	…	…	s_i	u_i
$\frac{\partial C(\bar{x})}{\partial x_{n1}}$	…	…	…	$\frac{\partial C(\bar{x})}{\partial x_{nm}}$	s_n	u_n
d_1	…	d_j	…	d_m		
v_1	…	v_j	…	v_m		

where $\bar{x}$ is the current basic solution.
The Lagrange function for the NTP is formulated as
$z(x, \lambda, w) = C(x) + w(b - Ax) - \lambda x$ … (5)
where λ and w are Lagrange multipliers and

$$\lambda \in \mathbb{R}_{+}^{nm} U(0)$$

$$w \in \mathbb{R}^{n+m}$$

The optimal point $\bar{x}$ should satisfy the KKT conditions:

$$\nabla z = \nabla C(\bar{x}) - w^T A - \lambda = 0$$
$$\lambda \bar{x} = 0$$
$$\lambda \geq 0$$
$$\bar{x} \geq 0$$

Specifically for each cell (i, j) we have

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{x})}{\partial x_{ij}} - (u, v)(e_i, e_{n+j}) - \lambda_{ij} = 0 \quad \dots(6)$$

$$\lambda_{ij} x_{ij} = 0$$
$$x_{ij} \geq 0$$
$$\lambda_{ij} \geq 0$$

where k = 1 … nm and w = (u, v) = (u_1, u_2, …, u_n, v_1, …, v_m), $e_k \in \mathbb{R}^{m+n}$ is a vector of zeros except at position k which is 1.
From the conditions (3.6) and $\lambda_k \geq 0$, we get,

$$\frac{\partial z}{\partial x_{ij}} = \frac{\partial C(\bar{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0 \qquad (7)$$

$$x_{ij} \frac{\partial z}{\partial x_{ij}} = x_{ij} \left(\frac{\partial C(\bar{x})}{\partial x_{ij}} - (u_i + v_j) \right) = 0 \quad \dots (8)$$

$$x_{ij} \geq 0$$

General solution procedure for the NTP

- **Initialization**

Find an initial basic feasible solution $\bar{x}$

- **Iteration**

Step I: if $\bar{x}$ is KKT point, stop. Otherwise go to the next step.
Step II: Find the new feasible solution that improves the cost function and go to step 1.

TRANSPORTATION PROBLEM WITH CONCAVE COST FUNCTIONS

For large distributions, volume discount may be available sometimes. In this case the cost function of the transportation problem generally takes concave structure for it is separable and the marginal cost (cost per unit commodity distributed) decreases with increase of the amount of distribution; and increasing, because of the total cost increase per addition of unit commodity distributed. The discount

1. May be either directly related to the unit commodity.
2. Or have the same rate for some amount.

Case 1: If the discount is directly related to the unit commodity the resulting cost function will be continues and have continues first partial derivatives. Nonlinear programming formulation of such a problem is given by

$$\begin{aligned} &\min \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij}(x_{ij}) \\ &\text{s.t} \sum_{j=1}^{m} x_{ij} = s_i \quad j = 1, 2, \dots, m \\ &\sum_{i=1}^{n} x_{ij} = d_j \quad i = 1, 2, \dots, n; \end{aligned} \qquad \dots(9)$$

Where $C_{ij} : \mathbb{R} \to \mathbb{R}$
Now before going to look for an optimal solution let's state an important theorem:

Theorem 3.3.1: Let f be concave and continues function and P be a non empty compact polyhedral set. Then the optimal solution to the problem min f(x), $x \in P$ exists and can be found at an extreme point of P.

Proof:
Let E = (x_1, x_2, …, x_k, …, x_n) be the set of extreme points of P, and $x_k \in E$ such that $f(x_k) = \min\{f(x_1): i = 1, \dots, n\}$. Now since P is compact and f is continuous, f attains its minimum in P, call it $\bar{x}$,
If $\bar{x}$ is extreme point, we are done. Otherwise, we have that,

$$\bar{x} = \sum_{i=1}^{n} \lambda_i x_i \qquad \sum_{i=1}^{n} \lambda_i = 1; \ \lambda_i > 0$$

where x_1, x_2, …, x_n are extreme points of P.
Then by concavity of f it follows that,

$$f(\bar{x}) = f\left(\sum_{i=1}^{n}\lambda_i x_i\right) \geq \sum_{i=1}^{n}\lambda_i f(x_i) \geq f(x_k)\sum_{i=1}^{n}\lambda_i$$

$\Rightarrow \quad f(\bar{x}) \geq f(x_k)$ (Since for each i = 1, …, n; $f(x_k) \leq f(x_i)$ and $\sum_{i=1}^{n}\lambda_i = 1$)

Since $\bar{x}$ is minimize, in addition we have,

$f(\bar{x}) \leq f(x_k)$

The above two relations imply

$f(\bar{x}) = f(x_k)$

This completes the proof.

Solution Procedure

Because of the above theorem, it suffices to consider only the extreme points to find the minimum; the following is the procedure.

After we find the initial basic feasible solution (they are n + m – 1 in number), let $\bar{x}$ be the basic solution we have in the current iteration.

Next let's decompose our $\bar{x}$ to ($\bar{x}_B$, $\bar{x}_N$) where $\bar{x}_B$ and $\bar{x}_N$ are the basic and nonbasic variables respectively. Since $\bar{x}_B > 0$, the complementary slackness condition given in (3.8) gives as m + n – 1 equations;

$$\frac{\partial z}{\partial x_{Bij}} = \frac{\partial C(\bar{x})}{\partial x_{Bij}} - (u_i - v_j) = 0 \quad \ldots (10)$$

From the above relation we can determine the values of u_i and v_j by assigning one of u_i's the value zero for we have m + n variables, u_i and v_j. Then we calculate $\frac{\partial z}{\partial x_{ij}}$ for the non basic variables x_{ij}. Since all x_{ij} are zero at the extreme, the complementary slackness condition is satisfied. Therefore if equation (3.2) is satisfied for all non basic variables x_{ij}, $\bar{x}$ is a KKT point. Otherwise, if

$$\frac{\partial z}{\partial x_{ij}} - (u_i + v_j) < 0 \qquad (11)$$

We will move to look for better basic solution such that all the constraints (feasibility conditions) are satisfied. We do this by using the same procedure as the transportation simplex algorithm.

THE TRANSPORTATION CONCAVE SIMPLEX ALGORITHM (TCS)

Initialization

Find the initial basic feasible solution using some rule like west corner rule.

Iteration

Step 1: Determine the values of u_i and v_i from the equation,

$$\frac{\partial C(\bar{x})}{\partial x_{Bij}} - (u_i + v_j) = 0 \qquad (12)$$

Where x_{Bij} are the basic variables.

Step 2: If

$$\frac{\partial C(\bar{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0 \qquad (13)$$

for all x_{ij} – non basic, stop, $\bar{x}$ is KKT point. Otherwise go to step 3.

Step 3: Calculate

$$\frac{\partial z}{\partial x_{rl}} = \left\{\frac{\partial C(\bar{x})}{\partial x_{ij}} - u_i - v_j\right\} \qquad (14)$$

x_{rl} will enter the basic. Allocate $x_{rl} = \theta$ where θ is found as in the linear transportation case.

Adjust the allocations so that the constraints are satisfied.

Determine the leaving variable say x_{Brk}, where x_{Brk} is the basic variable which comes to zero first while making the adjustment. Then find the new basic variables and go to step 1.

Finite Convergence of the Algorithm

The feasible set of our problem is a non empty polyhedral set. And by definition, a polyhedral set P is a set bounded with a finite number of hyper planes from which it follows that it possesses finite number of extreme points. In each step of the algorithm, we jump from one extreme point to another looking for a better feasible solution implying that the algorithm will terminate after a finite iteration. In addition since for all i and j, $x_{ij} \leq \max\{s_i, d_j\}$, P is bounded that guarantees the existence of minimum value.

Case 2: In the case when the volume discount is fixed for some amount of commodity, rather than varying with unit amount distributed, the transportation cost function will be piecewise linear concave yet increasing.

To avoid complication, assuming that to each combination of source and destination, the interval in which the marginal cost (cost per unit commodity) changes is the same, the cost of distributing x_{ij} units from source i to destination j is given by $C_{ij}(x_{ij})$, then the nonlinear programming formulation of the problem is given by

$$\left.\begin{aligned} &\min \sum_{i=1}^{n}\sum_{j=1}^{m} C_{ij}(x_{ij}) \\ &\text{s.t} \sum_{j=1}^{m} x_{ij} = s_i \end{aligned}\right\} \qquad (15)$$

$$\sum_{i=1}^{n} x_{ij} = d_j$$

$i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m$

where

$$C_{ij}(x_{ij}) = \begin{cases} C_{ij}^0(x_{ij}), 0 \le x_{ij} \le a_1 \\ C_{ij}^1(x_{ij}), a \le x_{ij} \le a_2 \\ \vdots \\ C_{ij}^l(x_{ij}), a_1 \le x_{ij} \le a_{l+1} \\ \vdots \\ C_{ij}^{k-1}(x_{ij}), a_{k-1} \le x_{ij} \le a_k \\ C_{ij}^k(x_{ij}), a_k \le x_{ij} \le b = \max\{s_i, d_j\} \end{cases}$$

And

1. $(0, a_1, \ldots, a_l, \ldots, a_{k-1}, a_k, b\}$ is the partition of the interval [0, b] into $k + 1$ sub intervals
2. Each C_{ij}^l is linear in the sub interval $[a_l; a_{l+1}]$

To solve this problem, as we can see from the structure of the cost function, it's impossible to directly apply the algorithm of the previous section for non differentiability of the total cost function hinders as to do so.

But, since the function, also, has a simple structure and differentiability fails
at discrete points, it can be easily approximated using differentiable functions like Chebshev, trigonometric or Legendre polynomials.

We choose to approximate it by the so called shifted Legendre polynomials. These set of Legendre polynomials say $\{p_0, p_1, \ldots, p_r\}$ is orthogonal in [0,1] with respect to weight function $w(x) = 1$, where the inner product on C[0; 1] is defined by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx \text{ ; for all } f, g \in C$$

[0; 1];
where C[0; 1] is the space of continuous functions on [0; 1]
The first four of them are,

$$p_0(x) = 1$$
$$p_1(x) = 2x - 1$$
$$p_2(x) = 6x^2 - 6x + 1$$
$$p_3(x) = 20x^3 - 30x^2 + 12x - 1$$

and the others can be obtained from

$$pr(x) = \frac{1}{2^r!}\frac{d^r}{dx^2}[(x^2 - 1)^r]$$

Then, the space spanned by $\{p_0, p_1, \ldots, p_r\}$ is a subspace of C[0:1]. Hence, given any $f(x) \in C[0; 1]$, we can find a unique least square approximation of *f* in the subspace. Note that every element of the subspace spanned $\{p_0, p_1, \ldots, p_r\}$ is at least twice differentiable.

The least square approximation of any function f(x) with r of these polynomials in [0; 1] is given by,

$$\bar{f}(x) = a_0p_0(x) + a_1p_1(x) + \ldots + a_ip_i(x) \ldots + a_rp_r(x) \quad \ldots \quad (16)$$

where

$$a_i = \frac{\int_0^1 p_i f(x)dx}{\int_0^1 [p_i(x)]^2 dx}; \quad i = 0; 1, \ldots, r$$

To approximate our functions $C_{ij}(x_{ij})$,in the same manner, we define a one to one correspondence between [0,b] to [0,1] by

$$g : [0, b] \to [0, 1] \quad (17)$$
$$g(x_{ij}) = \frac{1}{b}x_{ij}$$

That is, we substitute x_{ij} by $\frac{1}{b}x_{ij}$ so that it's domain will be [0,1] then we have,

$$C_{ij}(x_{ij}) \to \hat{C}_{ij}(x_{ij}) = C_{ij}\left(\frac{1}{b}x_{ij}\right) = \begin{cases} C_{ij}^0\left(\frac{1}{b}x_{ij}\right), 0 \le x_{ij} \le \frac{a_1}{b} \\ C_{ij}^1\left(\frac{1}{b}x_{ij}\right), \frac{a_1}{b} \le x_{ij} \le \frac{a_2}{b} \\ \vdots \\ C_{ij}^k\left(\frac{1}{b}x_{ij}\right), \frac{a_k}{b} \le x_{ij} \le 1 \end{cases} \quad (18)$$

Now, after approximating $\hat{C}_{ij}x_{ij}$ by the shifted Legendre polynomials on
[0, 1], assume we have found it's best approximation $\tilde{C}_{ij}(x_{ij})$.

Then, substituting back the x_{ij} in $\bar{C}_{ij}$ by bx_{ij} gives us the approximation to
$C_{ij}(x_{ij})$ over [0, b]. Therefore the best approximation of $C_{ij}(x_{ij})$ over [0, b]
will be $\bar{C}_{ij}(x_{ij}) = \tilde{C}_{ij}(bx_{ij})$
which has continuous derivatives.
Consequently, we solve the problem

$$\left.\begin{aligned} \min \sum_{i=1}^{n}\sum_{j=1}^{m} \bar{C}_{ij}(x_{ij}) = \sum_{l=0}^{2}\sum_{i=1}^{n}\sum_{j=1}^{m} a_l p_l(x_{ij}) \\ \text{s.t} \sum_{j=1}^{m} x_{ij} = \frac{s_i}{b} \\ \sum_{i=1}^{n} x_{ij} = \frac{d_j}{b} \\ i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, m \end{aligned}\right\} \quad (19)$$

using exactly the same procedure as the previous case.

CONVEX TRANSPORTATION PROBLEM

This case may arise when the objective function is composed of not only the unit transportation cost but also of production cost related to each commodity [Shetty; 1959]. Or in the case when the distance from each source to each destination is not fixed.

The problem can be formulated as

$$\left.\begin{array}{l}\min C(x)\\ \text{s.t } Ax = b\\ \\ x \geq 0\end{array}\right\} \qquad (20)$$

where C(x) is convex, continuous and has continuous first order partial derivatives.

THE CONVEX SIMPLEX SOLUTION PROCEDURE FOR TRANSPORTATION PROBLEM

In the case when the cost function is convex, the minimum point may not be attained necessarily at an extreme; it may be found before reaching a boundary of the feasible set.

What precisely happens is that there may be non basic variable with positive allocation while none of the basis is driven to zero.

To solve this problem, we use the idea of the convex simplex algorithm of Zangwill [1967] which was originally designed to take care of convex and pseudo convex problem with linear constraints. Actually the original procedure is used to look for a local optimal solution for any other linearly constrained programming problem. We use the special structure of transportation problem in the procedure so as to make it efficient for our particular problem.

The method reduces to the ordinary transportation simplex algorithm whenever the objective is linear, to the method of Beal when it is quadratic and to the above concave simplex procedure when the function is concave.

We partition the variable $x = (x_{11}, \ldots, x_{nm})$ to (x_B, x_N) where x_B is $n + m - 1$ component vector of basic variables and x_N is $(nm - (n + m - 1))$ component vector of nonbasic variables, corresponding to the $(n + m - 1) \times (n + m - 1)$ basic sub matrix and $(n + m - 1)\times((nm - (n + - 1))$ non basic sub matrix of A.

Suppose we have the initial basic feasible solution $\overline{x}^o$.

In the procedure what we do is to find a mechanism in which non optimal basic solution $\overline{x}$ at a given iteration is improved until it satisfies the KKT conditions which are also sufficient conditions for convex transportation problem, i.e. until for each cell we have;

$$x_{ij}\left(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)\right) = 0 \qquad (21)$$

and $$\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0 \qquad (22)$$

Since we have each basic variable $x_{Bij} > 0$, the above complementary slackness condition implies that for each basic cell, we must have

$$\frac{\partial f(\overline{x})}{\partial x_{Bij}} - (u_i + v_j) = 0$$

x_{Bij} - basic variable.

Since we have $n + m - 1$ of such equations, by letting $u_1 = 0$ we obtain all the values of u_i and v_j as we have done exactly for the concave and linear cases.

Now for a non basic cell, at a feasible iterate point $\overline{x}$, we may have:

i.

$$\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) > 0;$$

$$x_{ij}\left(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)\right) > 0 \ldots (23)$$

ii.

$$\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) < 0;$$

$$x_{ij}\left(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)\right) < 0 \ \ldots \ (24)$$

iii.

$$\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) < 0;$$

$$x_{ij}\left(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)\right) = 0 \ \ldots \ (25)$$

Or for all non basic x_{ij}, we may have;

iv.

$$\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j) \geq 0;$$

$$x_{ij}\left(\frac{\partial f(\overline{x})}{\partial x_{ij}} - (u_i + v_j)\right) = 0 \ \ldots \ (26)$$

From the KKT conditions given earlier, the last case occurs when $\overline{x}$ is optimal.

But if the solution $\overline{x}$ falls on either of the other three, it must be improved as follows.

Let IJ = {ij : x_{ij} is non basic variable} and suppose that we are in the kth iteration.

We first begin by computing;

$$\frac{\partial z}{\partial x_{rl}} = \min\left\{\frac{\partial f(\bar{x}^k)}{\partial x_{ij}} - u_i - v_j\right\} ij \in IJ \ldots (27)$$

$$x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) = \max\left\{x_{ij}\left(\frac{\partial f(\bar{x}^k)}{\partial x_{ij}} - u_i - v_j\right)\right\}$$
$$ij \in IJ \quad \ldots (28)$$

Here we don't want to improve (decrease) a positive - valued non basic variable x_{ij} unless its partial derivative is positive. Therefore we only focus on positive values of the product $\frac{\partial z}{\partial x_{ij}} x_{ij}$

Now the variables to be adjusted are selected as;

Case 1: If $\frac{\partial z}{\partial x_{rl}} \geq 0$ and $x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) > 0 \ldots (29)$

Decrease x_{st} by the value θ using the transportation table as in the
linear and concave cases.

Let $y^k = (y_{11}^k, y_{12}^k, \ldots, y_{nm}^k)$ be the value of $\bar{x}^k = (\bar{x}_{11}^k, \ldots, \bar{x}_{nm}^k)$ after making the necessary adjustment by adding and subtracting θ in the loop containing x_{st} so that all the constraints are satisfied. By doing so, either x_{st} itself or a basic variable say x_{Bst} will be driven to zero.

Now y^k may not be the next iterate point; since the function is convex, a better point could be found before reaching y^k to check this, we solve problem;

$f(\bar{x}^{k+1}) = \min\{f(\lambda\bar{x}^k + (1-\lambda)y^k : 0 \leq \lambda \leq 1 \quad (30)$

and get $\bar{x}^{k+1} = \lambda\bar{x}^k + (1-\lambda)y^k$ where $\bar{\lambda}$ is the optimal solution of (30).

Before the next iteration,

If $\bar{x}^{k+1} = y^k$ and if a basic variable became zero during the adjustment made, we change the basis.

If $\bar{x}^{k+1} \neq y^k$ or if $\bar{x}^{k+1} = y^k$ and x_{st} is driven to zero, we don't change the basis by substituting the leaving basic variable by x_{st}.

Case 2: If

$$\frac{\partial z}{\partial x_{rl}} < 0 \text{ and } x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) \leq 0 \quad \ldots (31)$$

In this case the value of x_{rl} should be increased by θ and then we find y^k, where θ and y^k are defined as in the **case 1.**

Note that: as we increase the value of x_{rl} one of the basic variables, say, x_{Bt} will be driven to zero, and this is the exit criteria of the linear and concave transportation simplex algorithm and y^k would have been the next iterate point of the procedure.

But now after solving for $\bar{x}^{k+1}$ from (30), before going to the next iteration, we will have the following possibilities.

If $\bar{x}^{k+1} = y^k$, we change the former basis substitute x_{Bt} by x_{rl}

If $\bar{x}^{k+1} \neq y^k$, we do not change the basis.

All the basic variables outside of the loop will remain unchanged.

Case 3: If $\frac{\partial z}{\partial x_{rl}} < 0$ and $x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) > 0 \ldots (32)$

In this case either we decrease x_{st} as in the case 1 or increase x_{rl} according to case 2.

THE TRANSPORTATION CONVEX SIMPLEX ALGORITHM

Now we write the formal algorithm* for solving the convex transportation problem.

Initialization

Find the initial basic feasible solution.

Iteration

Step 1: Determine all u_i and v_j from

$$\frac{\partial f(\bar{x}^k)}{\partial x_{Bij}} - u_i - v_j = 0 \quad (33)$$

for each basic cell

Step 2: For each non basic cell, calculate;

$$\frac{\partial z}{\partial x_{rl}} = \min\left\{\frac{\partial f(\bar{x}^k)}{\partial x_{ij}} - u_i - v_j\right\} \quad (34)$$

$$x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) = \min\left\{x_{ij}\left(\frac{\partial f(\bar{x}^k)}{\partial x_{ij}} - u_i - v_j\right)\right\} \quad (35)$$

If $\frac{\partial z}{\partial x_{rl}} < 0$ and $x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) = 0 \quad (36)$

Stop. Otherwise go to step 3.

Step 3: Determine the non basic variable to change. Decrease x_{st} according to case 1 if

$$\frac{\partial z}{\partial x_{rl}} \geq 0 \text{ and } x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) > 0 \ldots (37)$$

Increase x_{rl} according to case 2 if

$$\frac{\partial z}{\partial x_{rk}} < 0 \text{ and } x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) > 0 \quad \ldots (38)$$

Either increase x_{rl} or decrease x_{st} if

$$\frac{\partial z}{\partial x_{rk}} < 0 \text{ and } x_{st}\left(\frac{\partial z}{\partial x_{st}}\right) > 0 \quad \ldots (39)$$

Step 4: Find the values of y^k, by means of θ, and $\bar{x}^{k+1}$, from 3.5. If $y^k = \bar{x}^{k+1}$ and a basic variable is

driven to zero, change the basis. Otherwise do not change the basis.

$$\overline{X}^{k} = \overline{X}^{k+1}$$

go to step 1.

DATA ANALYSIS

In this session, we shall examine a practical application of the above solution procedures. Emphasis will be on the concave transportation problem. We shall examine the data obtained from the Port-Harcourt Flour mills company Limited; manipulate the data to suit our transportation problem.

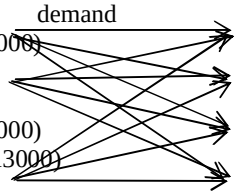
DATA AND ANALYSIS

The flour mills limited is a manufacturing company located in Port-Harcourt. The produce Golden Penny Flour (GPF), Golden Penny Semovita (GPS), Wheat Offals (WO) etc. These products are supplied to the following states (locations) Bayelsa, Onitsha, Portharcourt, Kano, Aba, Enugu etc. For the purpose of this thesis, only four (4) of these demand points will be considered; Enugu, Onitsha, Bayelsa and Aba. The estimated supply capacities of the three products, the demand requirements at the four sites (states) and the transportation cost per bag of each product are given below:

	Aba	Enugu	Onitsha	Bayelsa	Supply
GPF	8	20	16	10	17000
WO	6	14	16	12	11000
GPS	8	10	6	12	15000
Demand	12000	13000	11000	7000	43000

The problem is to determine how many bags of each product to be transported from the source to each destination on a monthly basis in order to minimize the total transportation cost.

A diagram of the different transportation routes with supply and demand figures is shown below.



Forming the transportation tableau. To form the transportation tableau, let:

i = product to be shipped

j = destination of each product

S_i = the capacity of source node i,

d_j = the demand of destination j,

x_{ij} = the total capacity from source i to destination

c_{ij} = the per unit cost of transporting commodity from source i to destination j.

p_m = percentage discount allowed for transporting from i to destination j.

If we suppose that discount is given on each bag transported from i to j, then the non linear transportation problem can be formulated as:

$$\min \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij}x_{ij}$$

S.t.
$$x_{11} + x_{12} + x_{13} + x_{14} = 17$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 11$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 15$$
$$x_{11} + x_{21} + x_{31} = 12$$
$$x_{12} + x_{22} + x_{32} = 13$$
$$x_{13} + x_{23} + x_{33} = 11$$
$$x_{14} + x_{24} + x_{34} = 7$$

where

$c_{11}x_{11} = 8x_{11} - p_{11}x_{11}^2$, $c_{23}x_{23} = 16x_{23} - p_{23}x_{23}^2$

$c_{12}x_{12} = 20x_{12} - p_{12}x_{12}^2$, $c_{24}x_{24} = 12x_{24} - p_{24}x_{24}^2$

$c_{13}x_{13} = 16x_{13} - p_{13}x_{13}^2$, $c_{31}x_{31} = 8x_{31} - p_{31}x_{31}^2$

$c_{14}x_{14} = 10x_{14} - p_{14}x_{14}^2$, $c_{32}x_{32} = 10x_{32} - p_{32}x_{32}^2$

$c_{21}x_{21} = 6x_{21} - p_{21}x_{21}^2$, $c_{33}x_{33} = 6x_{33} - p_{33}x_{33}^2$

$c_{22}x_{22} = 14x_{22} - p_{22}x_{22}^2$, $c_{34}x_{34} = 12x_{34} - p_{34}x_{34}^2$

If we allow the following discounts on each transported product i from the source to each of the destinations,

$(p_{11}, p_{12}, p_{13}, p_{14}, p_{21}, p_{22}, p_{23}, p_{24}, p_{31}, p_{32}, p_{33}, p_{34})$ = (0.02, 0.05, 0.03, 0.025, 0.01, 0.03, 0.035, 0.02, 0.025, 0.03, 0.015, 0.02)

Thus, the cost function c_{ij} can be expressed as

$c_{11}x_{11} = 8x_{11} - 0.02x_{11}^2$, $c_{23}x_{23} = 16x_{23} - 0.035x_{23}^2$

$c_{12}x_{12} = 20x_{12} - 0.05x_{12}^2$, $c_{24}x_{234} = 12x_{24} - 0.02x_{24}^2$

$c_{13}x_{13} = 16x_{13} - 0.03x_{13}^2$, $c_{31}x_{31} = 8x_{31} - 0.025x_{31}^2$

$c_{14}x_{14} = 10x_{14} - 0.025x_{14}^2$, $c_{32}x_{32} = 10x_{32} - 0.02x_{32}^2$

$c_{21}x_{21} = 6x_{21} - 0.01x_{21}^2$, $c_{33}x_{33} = 6x_{33} - 0.015x_{33}^2$

$c_{22}x_{22} = 14x_{21} - 0.03x_{22}^2$, $c_{34}x_{34} = 12x_{34} - 0.02x_{34}^2$

we then develop the tableau as below;

	Aba	Enugu	Onitsha	Bayelsa	S_i	u_i
GPF	8	20	16	10	17000	u_1
WO	6	14	16	12	11000	u_2
GPS	8	10	6	12	15000	u_3
d_j	12000	13000	11000	7000		
v_j	v_1	v_2	v_3	v_4		

Total supply = 43,000

Total demand = 43,000

Hence the tableau is balanced

Using the vogel's Approximation Method (VAM), we get the initial basic solution.

The solution tableau is as shown below

	Aba	Enugu	Onitsha	Bayelsa	S_i	u_i
GPF	8 [1]	20 [9]	16	10 [7]	17000	u_1
WO	6 [11]	14	16	12	11000	u_2
GPS	8	10 [4]	6 [11]	12	15000	u_3
d_j	12000	13000	11000	7000		
v_j	v_1	v_2	v_3	v_4		

$\overline{x}$ = (x_{B11}, x_{B12}, x_{B13}, x_{B14}, x_{B21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{B32}, x_{B33}, x_{34})
= (1, 9, 0, 7, 11, 0, 0, 0, 0, 4, 11, 0), in thousands.

The total transportation cost becomes
1000(8) + 9000(20) + 7000(10) + 11000(6) + 4000(10) + 11000(6)
= ₦430,000
The partial derivations at x are given as:

$$\frac{\partial f(\overline{x})}{\partial x_{11}} = 7.96;\ \frac{\partial f(\overline{x})}{\partial x_{12}} = 19.1;$$

$$\frac{\partial f(\overline{x})}{\partial x_{13}} = 16;\frac{\partial f(\overline{x})}{\partial x_{14}} = 9.65;$$

$$\frac{\partial f(\overline{x})}{\partial x_{21}} = 5.78;\ \frac{\partial f(\overline{x})}{\partial x_{22}} = 14;\frac{\partial f(\overline{x})}{\partial x_{23}} = 16;$$

$$\frac{\partial f(\overline{x})}{\partial x_{24}} = 12;\frac{\partial f(\overline{x})}{\partial x_{31}} = 8;\frac{\partial f(\overline{x})}{\partial x_{32}} = 9.84;$$

$$\frac{\partial f(\overline{x})}{\partial x_{33}} = 5.67;\ \frac{\partial f(\overline{x})}{\partial x_{34}} = 12$$

Now, we find

$$\frac{\partial z}{\partial x_{Bij}} = \frac{\partial f(\overline{x})}{\partial x_{Bij}} - u_i - v_j = 0$$

$$\Rightarrow \quad \frac{\partial f(\overline{x})}{\partial x_{Bij}} = u_i + v_j$$

Thus,

$u_1 + v_1 = 7.96$; $u_1 + v_2 = 19.1$; $u_1 + v_4 = 9.65$

$u_2 + v_1 = 5.78$; $u_3 + v_2 = 9.84$; $u_3 + v_3 = 5.67$

letting $u_1 = 0$, from the equations above; we have

$u_1 = 0$, $u_2 = -2.18$, $u_3 = -9.26$

$v_1 = 7.96$, $v_2 = 19.1$, $v_3 = 14.93$, $v_4 = 9.65$

Then, the reduced costs for the non-basic variables become

$$\frac{\partial z}{\partial x_{13}} = \frac{\partial f(\overline{x})}{\partial x_{13}} - u_1 - v_3 = 1.07$$

$$\frac{\partial z}{\partial x_{23}} = \frac{\partial f(\overline{x})}{\partial x_{23}} - u_2 - v_3 = 3.25$$

$$\frac{\partial z}{\partial x_{24}} = \frac{\partial f(\overline{x})}{\partial x_{24}} - u_2 - v_4 = 4.53$$

$$\frac{\partial z}{\partial x_{31}} = \frac{\partial f(\overline{x})}{\partial x_{31}} - u_3 - v_1 = 9.3$$

$$\frac{\partial z}{\partial x_{34}} = \frac{\partial f(\overline{x})}{\partial x_{34}} - u_3 - v_4 = 11.61$$

$$\min\left(\frac{\partial z}{\partial x_{ij}}\right) = \frac{\partial z}{\partial x_{22}} = -2.92$$

It is obvious that the presence of negative value for the reduced cost signifies non optimality; hence we readjust.

Therefore x_{22} should enter the basis since it is the most negative reduced cost, after adjusting the values x_{12} left the basic.

	Aba	Enugu	Onitsha	Bayelsa	Supply	u_i
GPF	8 [1] +	20 [9] -	16 (1.07)	10 [7]	17000	0
WO	6 [11] -	14 + (-.92)	16 (3.25)	12 (4.53)	11000	-2.18
GPS	8 (9.3)	10 [4]	6 [11]	12 (11.61)	15000	-9.26
d_j	12000	13000	11000	7000		
v_j	7.96	19.1	14.93	9.65		

The basic variable with the least value among the corners having sign in the loop is the leaving variable. Hence, x_{12} with the least value of 9 is the leaving variables. Thus, we increase the corners with + sign by 9, reduce the ones with – sign by 9. The adjusted tableau becomes:

	Aba	Enugu	Onitsha	Bayelsa	Supply	u_i
GPF	8 [10]	20	16	10 [7]	17000	0
WO	6 [2]	14 [9]	16	12	11000	-2.18
GPS	8	10 [4]	6 [11]	12	15000	-9.26
d_j	12000	13000	11000	7000		
v_j	7.96	19.1	14.93	9.65		

The reduced costs for the non-basic ones at a basic feasible point.

$\overline{x}^2$= (x_{B11}, x_{12}, x_{13}, x_{B14}, x_{B21}, x_{B22}, x_{23}, x_{24}, x_{31}, x_{B32}, x_{B33}, x_{34})

= (10, 0, 1, 7, 2, 9, 0, 0, 0, 4, 11, 0)

will be;

$$\frac{\partial z}{\partial x_{22}} = \frac{\partial f(\overline{x})}{\partial x_{22}} - u_1 - v_2 = -2.92$$

$$\frac{\partial z}{\partial x_{12}} = \frac{\partial f(\overline{x})}{\partial x_{12}} - u_1 - v_2 = 0$$

$$\frac{\partial z}{\partial x_{13}} = \frac{\partial f(\overline{x})}{\partial x_{13}} - u_1 - v_3 = 1.07$$

$$\frac{\partial z}{\partial x_{23}} = \frac{\partial f(\bar{x})}{\partial x_{23}} - u_2 - v_3 = 3.25$$
$$\frac{\partial z}{\partial x_{24}} = \frac{\partial f(\bar{x})}{\partial x_{24}} - u_2 - v_4 = 4.53$$
$$\frac{\partial z}{\partial x_{31}} = \frac{\partial f(\bar{x})}{\partial x_{34}} - u_3 - v_1 = 9.3$$
$$\frac{\partial z}{\partial x_{34}} = \frac{\partial f(\bar{x})}{\partial x_{34}} - u_3 - v_4 = 11.61$$

All are non-negative, implying that $\bar{x}^2$ is a KKT point. Hence, an optimal solution to our problem.
Thus, the following allocation should be made: 10000 bags of GPF should be supplied to Aba, 7000 of the same product be supplied to Bayelsa. Allocate 2000 bags of WO to Aba, and 9000 bags of the same product to Enugu. Finally, allocate 4000 bags of GPS to Enugu and 11000 bags of the same product to Onitsha.

Total cost = 10000(8) + 7000(10) + 2000(6) + 9000(14) + 4000(10) + 11000(6) = ₦394,000

SUMMARY

In some occasion, there may be different ways to model a particular problem, but choosing the best approach reduces the complexity of the problem and time to solve. Since any programming problem with constraint matrix, structure the same as the transportation type problem, it can be regarded as a transportation type problem regardless of its physical meaning and because of its simple form, modeling such problems as transportation problem requires much less effort to solve then modeling it differently.

In this work, the nonlinear transportation problem is considered as a nonlinear programming problem and algorithms to solve this particular problem are given. The first algorithm is similar to that of the transportation simplex algorithm except for the nonlinearity assumption. The second algorithm is dependent on the simplex algorithm of Zangwill that we modified to use the special property of the coefficient matrix of the transportation problem so that we may take shortcuts to make problem solving simple.

CONCLUSION

In this research study, we were able to identify the problem confronting Port-Harcourt Flour Mills Company Plc as a transportation problem, formulate a mathematical model that represents the essence of the problem, identify the functional equations of the problem as well as solved the problem using the Karush-Kuha-Tucker (KKT) optimality condition for nonlinear programming problem. From the solution obtained, we were able to determine the minimum total cost of transportation as ₦394,000. Hence, the analysis revealed that allocation should be made as follows: 10000 bags of Golden Penny Flour should be supplied to Aba, 7000 of the same product be supplied to Bayelsa. Allocate 2000 bags of Wheat Offal to Aba and 9000 bags of the same product to Enugu and 11000 bags of the same product to Onitsha.

We then conclude that given discounts on cost of transportation could lead to increased productivity of producers. This is as a result of the fact that wholesalers and retailers, will have to pay less on transport for buying in large quantities, subsequently, consumers will buy at lower cost comparatively.

RECOMMENDATION

The algorithms used in this research work are not compared to any other previous algorithms; therefore, in the future, work should be done to:

1. Measure the efficiency of the algorithm
2. Check how near the solution of the approximated problem of the piecewise nonlinear transportation problem is to the optimal solution of the original problem.
3. To implement the algorithm to complex real life problems.

REFFERENCES

Abad, P. L. (1988). Determining optimal selling price and lot-size when the supplier offers all-unit Quantity Discounts, Decision Science, page 19, 622-634.

Adby P.R and Dempster M.A.H. (1976). Introduction to Optimization Methods.

Arcelus, F.J.; J. Bhadury and Srinivasan G (1996). Buyer-seller quantity discount strategies under profit maximizing objectives: A game-theoretical analysis, page 49-72.

Aucomp, D.C. and P.J. Kuzdrall (1986). Lot-sizes for one-time-only sales, Journal Operation Research Society, 37, 79-86.

Baker, R.C. (1976). Inventory policy for items on sale during regular replenishment, PIM 17, 55-63.

Banerjee, A. (1986). On a quantity discount-pricing model to increase vendor profits, Management Science. 32, 1513-1517.

Barman, S. and R.J. Tersine (1995). Economic purchasing strategies for temporary price discounts, European Journal of Operations Research, 80, 328-342.

Bastian, M. (1992). A perfect lot-tree procedure for the discounted dynamic lot-size problem with speculation, Naval Research Logistics, 30, 651-668.

Bazaraa M.S. (1993). Nonlinear programming: theory and algorithms, John Wiley and Sons, Inc, Sec. Ed.77

Benton, W.C. and D.C. Whybark (1982). Material requirements planning (MRP) and purchase discounts, journal of mathematics, page 137-143.

Benton, W.C. and S. PARK (1996). A classification of literature on determining the lot-size under quantity discounts, European journal of Operations Research, 92, 219-238.

Benton. W.C. (1991). Quality discount decisions under conditions of multiple items, multiple suppliers and resource limitations, IJPR, 29(10), 1953-1961.

Caputo A.C. (2006). A genetic approach for freight transportation planning, Industrial Management and Data Systems, Vol. 106 No. 5, page 719-738

Efroymson M.A. and Ray, T.L. (1966). Branch-Bound Algorithm for Plant Location, Operations Research, Vol 14, 361-368

Ellwein, L.B. (August, 1970). Fixed Charge Location-Allocation Problems with Capacity and Configuration Constraints, Technical Report"No.70-2, Department of Industrial Engineering, Stanford University,

Falk J.E. and Soland R.M. (1972). An Algorithm for Separable Nonconvex Programming Problems, Management Science Vol. 15, page 550-569

Frederick S., Hiller and Gerald J. Lieberman (1990). Introduction to mathematical programming, McGraw-Hill, Incopration, Second Edition.

Giddings A.P., Bauket T.G. and Moore J.T. (2001). Optimality analysis of facility Optimality analysis of facility location problems using response surface methodology, International Journal of Physical Distribution and Logistics Management, Vol. 31 No.1 page 38-52. MCB University.

Gray (1970). Exact Solution of the Site Selection Problem by Miced Integer Programming, Applied Mathematical Programming Techniques ,E.M.L. Beale(ed.), American Elsevier Publishing Co., New York.

Hitchcock F.L. (1941). The Distribution of a Product from Several Sources to Numerous Localities, journal of Mathematics and Physics Vol.20, page 224-230.78 Iie Trans., 20, 292-294. (1994). Optimal prices and order quantities when temporary price discounts result in increase in demand, European Journal of Operation Research. 72, 52-71.

Inyama S.C. (2007): Operation Research: Introduction. Supreme Publisher, Owerri.
Karak P. M. (2001). Linear Programming and Theory of Games, New central book agency (P) Ltd.

Khumwala B.M., (1972). An Ancient Branch and Bound Algorithm for the Warehouse Location Problem, Management Science. Vol.18, 718-731.

Koopmans T.C. (1947). Optimum Utilization of the Transportation Systems," International Statistical Conference, Washington, D.C., Vol.5.

Lau, H.C.W.; Chan, T.M.; Tsui, W.T.; Chan, F.T.S.; Ho, G.T.S and Choy K.L. (2009): A fuzzy guided multi-objective evolutionary algorithm model for solving traportation problem. Expert System with Applications: An International Journal. Vol. 36, pp. 8255 – 8268.

Letcher, R.F. (1969). Practical Methods of Optimization, University of Dundee, Scotland, U.K.,

Lohgaonkar, M.H. and Bajaj, V.H. (2010): Fuzzy approach to solve multi-objective capacitated transportation problem, International Journal of Bioinformatics Research. Vol. 2, pp. 10 – 14.

Marks D.H. (1980). Facility Location and Routing Models in Solid Waste Collection Systems, Ph.D. Thesis, The Johns Hopkins University.

Mohamed A. (1996). An iterative procedure for solving the uncapacitated production distribution problem under concave cost function, International Journal of operations and management, Vol. 16. No. 3 page 18-27.

Murtagh B.A., Sims J.W. (1995). Improved modeling of physical distribution, International Journal of Physical Distribution and Logistics Management, Vol. 25 No.8 page. 47-52, MCB University.

Reep J. and Leaengood S. (June 2002). Transportation problem: A Special Case of Linear Programming Problems, Operations Research Society of America.

Sa G. (1969). Branch- and- Bound and Approximate Solutions to the Capacitated Plant Location Problem, Operations Research, Vol. 17, page 1016-111.

Shafaa A; Goyal S.K. (April 1988). Resolution of Degeneracy in Transportation Problems, Operations Research Vol.39, No.4 411-413.

Shetty C.M. (1959). A Solution to the Transportation Problem with Nonlinear Costs," Operation Research. Vol. 7. No.5. page 571-580.

Simons (March, 1962). Nonlinear Programming for Operation research, Society for Industrial and Applied Mathematics, Vol.10, No.1.

Soland R.M. (1974). Optimal Facility Location with Concave Costs" Ops. Res. Vol. 22, No.2. page 373-382.

Spielberg K, (1967). An Algorithm for the Simple Plant Location Problem with Some Side conditions Operations Research, Vol.17, page 85-111.79

Stewart T.J. and H.W. Ithmann (1968). Two stage Optimization in Transportation Problem.

Stringer J. and Haley K.B, (1957). The Application of Linear Programming to Large-scale Transportation Problem, Process First International Conference on Operation Research, Oxford

Taha, H.A. (1992): Operation Research: An Introduction, 5th ed. (Macmillan, New York.

Vidale M.L. (1956). A Graphical Solution to the Transportation Problem," Operation Research, Vol. 4, page 193-203.

Williams A.C. (March, 1962). A Treatment of Transportation Problems by Decomposition, journal Society for Industrial and Applied Mathematics, Vol.10, No.1 page 35-48.

Zangwill W.I. (1967). The Convex Simplex Method, Theory Series.Vol.14, No.3, 221-238.

Zangiabadi, M and Maleki, H.R. (2007): Fuzzy goal programming for multi-objective transportation problems. Applied Mathematics and Computation, vol. 24, pp. 449 – 460.
http://www.blackwell-synergy.com.
http:// www.interscience.wiley.com.
http://www3.interscience.wiley.com.
http://jstor.org.
http://en.wikipedia.org/wiki/Transportation-problem.

APPENDIX (TORA SOFTWARE PACKAGE OUTPUT)

TRANSPORTATION MODEL (VOGEL'S METHOD)

Iteration 1	**ObjVal**	**430000.00**	**D1**	**D2**	**D3**	**D4**	**Supply**
	Name		v1=8.00	v2=20.00	v3=16.00	v4=10.00	
S1		u1=0.00	8.00 **1000** 0.00	20.00 **9000** 0.00	16.00 0.00	10.00 **7000** 0.00	**17000**
S2		u2= -2.00	6.00 **11000** 0.00	14.00 4.00	16.00 -2.00	12.00 -4.00	**11000**
S3		u3=-10.00	8.00 -10.00	10.00 **4000** 0.00	6.00 **11000** 0.00	12.00 -12.00	**15000**
	Demand		**12000**	**13000**	**11000**	**7000**	
Iteration 2	**ObjVal**	**394000.00**	**D1**	**D2**	**D3**	**D4**	**Supply**
	Name		v1=8.00	v2=20.00	v3=16.00	v4=10.00	
S1		u1=0.00	8.00 **10000** 0.00	20.00 -4.00	16.00 -4.00	10.00 **7000** 0.00	17000
S2		u2= -2.00	6.00 **2000** 0.00	14.00 **9000** 0.00	16.00 -6.00	12.00 -4.00	11000
S3		u3=-10.00	8.00 -6.00	10.00 **4000** 0.00	6.00 **11000** 0.00	12.00 -8.00	15000
	Demand supply		12000	13000	11000	7000	